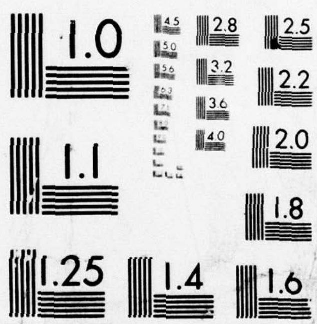
AD-A049 541   NORTHWESTERN UNIV EVANSTON ILL DEPT OF MECHANICAL E--ETC F/G 12/1
MULTICRITERIA OPTIMIZATION WITH UNCERTAINTY IN THE DYNAMICS.(U)
1977   W E SCHMITENDORF   AFOSR76-2923

UNCLASSIFIED   AFOSR-TR-78-0024   NL

| OF |
AD
A049541

END
DATE
FILMED
3-78
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MULTICRITERIA OPTIMIZATION WITH UNCERTAINTY IN THE DYNAMICS[1]

W. E. Schmitendorf
Northwestern University
Evanston, Illinois 60201

(2)

ABSTRACT

Optimal control problems with a vector performance index and uncertainty in the state equations are investigated. It is assumed that nature chooses the uncertainty, subject to bounds, to maximize the performance index which the controller attempts to minimize. Using Pareto optimality as the optimality criterion, sufficient conditions for an optimal solution are presented. These conditions also suggest a technique for determining the optimal control. The results are illustrated with an example.

1. INTRODUCTION

In the standard optimal control problem, the system to be controlled is modeled by ordinary differential equations and the cost is a scalar. In many problems, an exact description of the system is not available and disturbances are present in the differential equation model of the system. These disturbances may have a corrosive influence on the system performance if they are neglected. Another modification of the standard problem is that multiple criteria, rather than a single, scalar performance index, may be used to measure the performance of the system.

This paper treats multicriteria optimal control problems where there is uncertainty in the state equations. An optimality criterion is defined and then sufficient conditions for an optimal control are derived. In addition, it is shown how these results may be used to determine an optimal solution.

2. PROBLEM FORMULATION

Consider a control system described by the differential equations

$$\dot{x}(t) = f(x(t),u(t),v(t)) \tag{1}$$

where $x(t) \epsilon R^n$ is the state, $u(t) \epsilon R^{m_1}$ is the control and $v(t) \epsilon R^{m_2}$ is the disturbance. We assume that $f(\cdot,\cdot,\cdot)$ is a C^1 function. The time interval $[t_o,t_f]$ is specified as is the initial state.

$$x(t_o) = x_o \tag{2}$$

The disturbance $v(\cdot)$ is not known exactly; the only information available about $v(\cdot)$ is that it belongs to a known set $V \subset R^{m_2}$. Rather than modeling the system in a stochastic manner, we assume that nature chooses the disturbance in an antagonistic manner. Thus we assume that $v(\cdot)$ is chosen to maximize the criteria that the controller is attempting to minimize. This is a worst case design philosophy. Let

$$M_1 = \left\{u(\cdot):u(\cdot)\epsilon\ \mathcal{U} \text{ and } u(t)\epsilon U,\ t\epsilon[t_o,t_f]\right\}$$

where $\mathcal{U}$ is the set of piecewise continues functions from $[t_o,t_f] \to R^{m_1}$ and U is a given subset of R^{m_1}. Similarly, let

$$M_2 = \left\{v(\cdot):v(\cdot)\epsilon\ \mathcal{V} \text{ and } v(t)\epsilon V,\ t\epsilon[t_o,t_f]\right\}$$

where $\mathcal{V}$ is the set of piecewise continuous functions from $[t_o,t_f] \to R^{m_2}$. For $u(\cdot)\epsilon M_1$ and $v(\cdot)\epsilon M_2$, the solution of (1), (2) will be denoted by $x(\cdot)$. The pair $[u(\cdot),v(\cdot)]$ is <u>playable</u> if it generates a solution of (1), (2)

[1]This research was supported by AFOSR under Grant AFOSR-76-2923.

Dec 1973 in back

such that $x(t_f)\epsilon\theta$ where the target set θ is a given set in R^n.

For $u(\cdot)\epsilon M_1$, $N(u(\cdot))$ is the set of all $v(\cdot)\epsilon M_2$ such that $[u(\cdot),v(\cdot)]$ is playable. A control $u(\cdot)$ is <u>admissible</u> if $u(\cdot)\epsilon M_1$ and $N(u(\cdot))\neq \emptyset$.

For a playable pair $[u(\cdot),v(\cdot)]$, the cost is a vector with two components,

$$J_i(u(\cdot),v(\cdot);x(\cdot)) = g_i(x(t_f)) + \int_{t_o}^{t_f} L_i(x(t),u(t),v(t))dt, \quad i=1,2 \tag{3}$$

Here $x(\cdot)$ denotes the trajectory corresponding to $u(\cdot),v(\cdot)$ starting at (x_o,t_o). (The results presented below can easily be extended to the case where the cost is an K vector. For simplicity of presentation, we concentrate on the case K=2.) Our optimality criterion is an extension of the concept of Pareto optimality for problems where $v(\cdot)$ is not present.

<u>Definition</u> A control $u^*(\cdot)$ is Pareto optimal if, and only if, it is admissible and for all admissible $u(\cdot)$ either

$$\sup_{v(\cdot)\epsilon N(u^*(\cdot))} J_i(u^*(\cdot),v(\cdot);x(\cdot)) = \sup_{v(\cdot)\epsilon N(u(\cdot))} J_i(u(\cdot),v(\cdot);x(\cdot)), \quad i=1,2$$

or, for at least one $i\ \epsilon\{1,2\}$,

$$\sup_{v(\cdot)\epsilon N(u^*(\cdot))} J_i(u^*(\cdot),v(\cdot);x(\cdot)) < \sup_{v(\cdot)\epsilon N(u(\cdot))} J_i(u(\cdot),v(\cdot);x(\cdot))$$

When the cost has only one component, the definition becomes that of a minmax solution while, if there is no disturbance, the definition is the usual definition of Pareto optimality. If an admissible control, $u_1(\cdot)$, is not Pareto optimal, then there is another admissible control, $u_2(\cdot)$, such that

$$\sup_{v(\cdot)\epsilon N(u_2(\cdot))} J_i(u_2(\cdot),v(\cdot);x(\cdot)) \leq \sup_{v(\cdot)\epsilon N(u_1(\cdot))} J_i(u_1(\cdot),v(\cdot);x(\cdot)), \quad i=1,2$$

with the strict inequality holding for at least one i. Thus, if an admissible control is not Pareto optimal, there is another admissible control which guarantees that no component of the cost vector increases while at least one decreases.

Static versions of this problem are treated in [1-3]. In [4], a many player differential game problem with coalitions was studied and that problem can be interpreted as an optimal control problem with disturbances. There, however, in deriving sufficient conditions, it was assumed that a related two-player zero-sum game has a saddle point solution. We do not make that assumption here.

In the next section two sufficient conditions for Pareto optimal control are presented. These conditions are not directly working conditions, but only a step in that direction. In Sec. 4, we use the second of these conditions to obtain other conditions which may be used for determing Pareto optimal controls.

3. PRELIMINARY RESULTS

Sufficient conditions for Pareto optimality will be obtained in terms of the following two-player zero-sum game.

$$\hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) = \alpha_1 J_1(u(\cdot),v_1(\cdot);x_1(\cdot)) + \alpha_2 J_2(u(\cdot),v_2(\cdot);x_2(\cdot)) \tag{4}$$

$$\dot{x}_1(t) = f(x_1(t),u(t),v_1(t)), \qquad x_1(t_o) = x_o \tag{5}$$
$$\dot{x}_2(t) = f(x_2(t),u(t),v_2(t)), \qquad x_2(t_o) = x_o \tag{6}$$

Here $x_1(t),x_2(t)\epsilon R^n, v_1(\cdot),v_2(\cdot)\epsilon M_2$ and admissible $u(\cdot)$ are defined as in Sec. 2. One player controls $u(\cdot)$ while the other controls $v_1(\cdot)$ and $v_2(\cdot)$. Note that $x_i(\cdot)$ and $v_i(\cdot)$ only enter $\hat{J}$ through J_i while $u(\cdot)$enters through $J_1$and J_2. The time interval $[t_o,t_f]$is fixed and we require $x_1(t_f), x_2(t_f)\epsilon\theta$.

The controls $[u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)]$ are a saddle point solution for this game if they satisfy

$$\hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) \le \hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot);x_1^*(\cdot),x_2^*(\cdot))$$
$$\le \hat{J}(u(\cdot),v_1^*(\cdot),v_2^*(\cdot);x_1(\cdot),x_2(\cdot)) \tag{7}$$

for $v_1(\cdot),v_2(\cdot)\epsilon N(u^*(\cdot))$and for all $u(\cdot)$that are playable against $v_1^*(\cdot)$ and $v_2^*(\cdot)$. In Theorem 3.1,we assume all $u(\cdot)\epsilon M_1$ are playable against$(v_1^*(\cdot),v_2^*(\cdot))$.

<u>Theorem 3.1</u> If $[u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)]$ is a saddle point solution to the differential game (4)-(6) for some $\alpha_1,\alpha_2 > 0$, $\alpha_1+\alpha_2= 1$, then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

<u>Proof</u>. The saddle point inequalities (7) imply

$$\hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot);x_1^*(\cdot),x_2^*(\cdot)) = \sup_{v_1(\cdot),v_2(\cdot)\epsilon N(u^*(\cdot))} \hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot))$$

and that for any admissible $u(\cdot)$

$$\hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot);x_1^*(\cdot),x_2^*(\cdot)) \le \sup_{v_1(\cdot),v_2(\cdot)\epsilon N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot))$$

Thus

$$\sup_{v_1(\cdot),v_2(\cdot)\epsilon N(u^*(\cdot))} \hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) \le \sup_{v_1(\cdot),v_2(\cdot)\epsilon N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) \tag{8}$$

for all admissible $u(\cdot)$. Since $v_i(\cdot)$ and $x_i(\cdot)$ only affect J_i

$$\sup_{v_1(\cdot),v_2(\cdot)\epsilon N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) = \sum_{i=1}^{2} \alpha_i \sup_{v_i(\cdot)\epsilon N(u(\cdot))} J_i(u(\cdot),v_i(\cdot);x_i(\cdot)) \tag{9}$$

From (8), (9) it follows that

$$\sum_{i=1}^{2} \alpha_i \sup_{v_i(\cdot)\epsilon N(u^*(\cdot))} J_i(u^*(\cdot),v_i(\cdot);x_i(\cdot)) \le \sum_{i=1}^{2} \sup_{v_i(\cdot)\epsilon N(u(\cdot))} J_i(u(\cdot),v_i(\cdot);x_i(\cdot)) \tag{10}$$

Since $\alpha_1,\alpha_2 > 0$, (10) implies that $u^*(\cdot)$ is Pareto optimal. □

From this theorem, it is seen that Pareto optimal solutions can be found by determining open loop saddle point solutions for the two-player zero-sum game (4)-(6). As α_1 and α_2 vary, different saddle point solutions may be obtained. Each of these solutions has the Pareto optimal property. Techniques for finding open loop saddle point solutions can be found in [4-8]. Unfortunately, many of the differential games (4)-(6) which arise

from problems with a disturbance (1)-(3) do not have saddle point solutions. They do, however, have minmax solutions. An admissible control $u^*(\cdot)$ is a minmax solution to the game (4)-(6) if it satisfies for all admissible $u(\cdot)$

$$\sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot))$$
$$\leq \sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot))$$

Theorem 3.2 If $u^*(\cdot)$ is a minmax solution to the game (4)-(6) for some $\alpha_1,\alpha_2 > 0$, $\alpha_1+\alpha_2 = 1$, then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

Proof. Since $u^*(\cdot)$ is a minmax solution, it satisfies (8). The proof then follows in exactly the same manner as the proof of Theorem 3.1. □

Theorems similar to Theorem 3.1 and Theorem 3.2 were obtained for problems involving coalitions in a differential game [4]. There, use was made of Theorem 3.1 to find coalitive Pareto optimal controls. Here, we shall concentrate on Theorem 3.2 and use it, in conjunction with a sufficient condition for minmax control [9], to obtain sufficient conditions for a Pareto optimal solution to the problem with disturbances (1)-(3). Also a possible technique for determining Pareto optimal solutions, based on these sufficient conditions, will be discussed.

4. MAIN RESULT

In [8,9] there are sufficient conditions for a control to be a minmax control. Based on these results, as well as Theorem 3.2, we obtain sufficient conditions for a Pareto optimal control for the original problem (1)-(3). First we need some definitions. For $i=1,2$, the set $\mathcal{L}_i(u(\cdot))$ consists of the admissible disturbances that are playable against $u(\cdot)$ and maximize J_i.

$$\mathcal{L}_i(u(\cdot)) = \Big\{\hat{v}(\cdot):\hat{v}(\cdot)\in N(u(\cdot)) \text{ and } J_i(u(\cdot),\hat{v}(\cdot);x(\cdot)) = \sup_{v(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v(\cdot);x(\cdot))\Big\} \qquad i=1,2$$

Let $w(\cdot,\cdot)$ be a function from $R^n \times R^{m_1} \to R^{m_2}$ and, for $u(\cdot)\in M_1$, $q_w(\cdot)$ is the function defined by $q_w(t) = w(x(t),u(t))$ where $x(\cdot)$ is the solution of $\dot{x}(t) = f(x(t),u(t),w(x(t),u(t))),x(t_o) = x_o$. The functions $w(\cdot,\cdot)$ will be called response functions. The set of <u>admissible</u> <u>response</u> <u>function</u> is

$$\mathfrak{W} = \{w(\cdot,\cdot):q_w(\cdot)\in N(u(\cdot)) \text{ for all admissible } u(\cdot)\}$$

Let γ be a positive integer, $y^i(\cdot,\cdot)\in\mathfrak{W}$ $i=1,..,\gamma$, $z^i(\cdot,\cdot)\in\mathfrak{W}$, $i=1,..,\gamma$. Consider an optimal control problem with cost

$$K(u(\cdot),y^i(\cdot,\cdot),z^i(\cdot,\cdot);x_1^i(\cdot),x_2^i(\cdot))$$
$$= \sum_{i=1}^{\gamma} \mu_i \left\{\alpha_1 J_1(u(\cdot),y^i(\cdot,\cdot);x_1^i(\cdot)) + \alpha_2 J_2(u(\cdot),z^i(\cdot,\cdot);x_2^i(\cdot))\right\} \tag{11}$$

and state equations

$$\dot{x}_1^i(t) = f[x_1^i(t),u(t),y^i(x_1^i(t),u(t))],\quad x_1^i(t_o) = x_o \quad i=1,..,\gamma \tag{12a}$$

$$\dot{x}_2^i(t) = f[x_2^i(t),u(t),z^i(x_2^i(t),u(t))],\quad x_2^i(t_o) = x_o \quad i=1,..,\gamma \tag{12b}$$

The terminal conditions are $x_1^i(t_f)$, $x_2^i(t_f)\epsilon\theta$, $i=1,..,\gamma$.

In the following, we shall require $\alpha_1 > 0$, $\alpha_2 > 0$. Letting $\rho = \alpha_2/\alpha_1$, (11) can be rewritten as

$$K(u(\cdot),y^i(\cdot,\cdot),z^i(\cdot,\cdot);x_1^i(\cdot),x_2^i(\cdot))$$

$$=\sum_{i=1}^{\gamma}\mu_i J_1(u(\cdot),y^i(\cdot,\cdot);x_1^i(\cdot)) + \rho \sum_{i=1}^{\gamma}\mu_i J_2(u(\cdot),z^i(\cdot,\cdot);x_2^i(\cdot)) \qquad (11')$$

We now present a sufficient condition for a control $u^*(\cdot)$ to be a min-max control for the problem (4)-(6) and thus, from Theorem 3.2, a Pareto optimal control for the problem (1)-(3).

Theorem 4.1 Let $u^*(\cdot)$ be admissible. If there exists

i) a positive integer γ

ii) scalars $\mu_i > 0$, $i=1,..,\gamma$, $\sum_{i=1}^{\gamma}\mu_i = 1$ and a scalar $\rho > 0$

iii) admissible response $y^i(\cdot,\cdot),z^i(\cdot,\cdot)\epsilon\mathfrak{Y}$, $i=1,..,\gamma$

such that

a) $v_1^{i*}(\cdot)\epsilon\ \mathcal{L}_1(u^*(\cdot))$, $v_2^{i*}(\cdot)\epsilon\ \mathcal{L}_2(u^*(\cdot))$, $i=1,..,\gamma$

where $v_1^{i*}(t) = y^i(x_1^{i*}(t),u^*(t))$, $v_2^{i*}(t) = z^i(x_2^{i*}(t),u^*(t))$ and $x_1^{i*}(\cdot),x_2^{i*}(\cdot)$ are solutions of

$$\dot{x}_1^i(t) = f[x_1^i(t),u^*(t),y^i(x_1^i(t),u^*(t))],\quad x_1^i(t_o) = x_o \qquad i=1,..,\gamma$$

$$\dot{x}_2^i(t) = f[x_i^2(t),u^*(t),z^i(x_2^i(t),u^*(t))],\quad x_2^i(t_o) = x_o \qquad i=1,..,\gamma$$

b) $u^*(\cdot)$ is an optimal control for the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot),$ $z^i(\cdot,\cdot))$ then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

Proof. Consider any admissible $\hat{u}(\cdot)$. Since $u^*(\cdot)$ is a solution of the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot)),\ z^i(\cdot,\cdot))$

$$\sum_{i=1}^{\gamma}\mu_i J_1(u^*(\cdot),v_1^{i*}(\cdot);\ x_1^{i*}(\cdot)) + \rho \sum_{i=1}^{\gamma}\mu_i J_2(u^*(\cdot),v_2^{i*}(\cdot));\ x_2^{i*}(\cdot))$$

$$\le \sum_{i=1}^{\gamma}\mu_i J_1(\hat{u}(\cdot);q_{\hat{u}}^i(\cdot),\hat{x}_1^i(\cdot)) + \rho \sum_{i=1}^{\gamma}\mu_i J_2(u^*(\cdot),p_{\hat{u}}^i(\cdot);\hat{x}_2^i(\cdot))$$

$$q_{\hat{u}}^i(t) = y^i(\hat{x}_1^i(t),\hat{u}(t)) \qquad i=1,..,\gamma$$

$$p_{\hat{u}}^i(t) = z^i(\hat{x}_2^i(t),\hat{u}(t)) \qquad i=1,..,\gamma$$

Since $\rho > 0$, $\mu_i > 0$ $i=1,..,\gamma$, either

$$J_1(u^*(\cdot),v_1^{i*}(\cdot);x_1^{i*}(\cdot)) = J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot);\hat{x}_1^i(\cdot)) \qquad i=1,..,\gamma$$

and $J_2(u^*(\cdot),v_2^{i*}(\cdot);x_2^{i*}(\cdot)) = J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot);\hat{x}_2^i(\cdot)) \qquad i=1,..,\gamma \qquad (13)$

or

$$J_1(u^*(\cdot),v_1^{i*}(\cdot);x_1^{i*}(\cdot)) < J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot);x_1^i(\cdot) \text{ for some } i\epsilon\{1,..,\gamma\} \qquad (14)$$

or

$$J_2(u^*(\cdot),v_2^{i*}(\cdot);x_2^{i*}(\cdot)) < J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot);x_2^i(\cdot)) \text{ for some } i\epsilon\{1,..,\gamma\} \quad (15)$$

From (a)

$$\sup_{v\epsilon N(u^*(\cdot))} J_1(u^*(\cdot),v(\cdot);x(\cdot)) = J_1(u^*(\cdot),v_1^{i*}(\cdot);x_1^{i*}(\cdot)) \quad i=1,..,\gamma \quad (16)$$

$$\sup_{v\epsilon N(u^*(\cdot))} J_2(u^*(\cdot),v(\cdot);x(\cdot)) = J_2(u^*(\cdot),v_2^{i*}(\cdot);x_2^{i*}(\cdot)) \quad i=1,..,\gamma \quad (17)$$

Also,

$$J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot);\hat{x}_1^i(\cdot)) \le \sup_{v(\cdot)\epsilon N(\hat{u}(\cdot))} J_1(\hat{u}(\cdot),v(\cdot);x(\cdot)) \quad (18)$$

$$J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot);\hat{x}_2^i(\cdot)) \le \sup_{v(\cdot)\epsilon N(u(\cdot))} J_2(\hat{u}(\cdot),v(\cdot),x(\cdot)) \quad (19)$$

Using (16)-(19), each of the three cases (13), (14), and (15) imply $u^*(\cdot)$ is Pareto optimal.

5. DISCUSSION

For a given initial state and time, one can proceed as follows to obtain Pareto optimal solutions. Form the two-player zero-sum game (4)-(6). For those values of α_1 and α_2 with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$ for which there is an open loop saddle point, the corresponding $u(\cdot)$ is, according to Theorem 3.1, a Pareto optimal solution.

If there does not exist a saddle point solution, we must resort to Theorem 4.1. Choose a real integer $\gamma \ge 2$ and functions $y^i(\cdot,\cdot)$, $z^i(\cdot,\cdot)$, $i=1,..,\gamma$. Then determine the optimal control for the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot),z^i(\cdot,\cdot))$ in terms of ρ and μ_i, i.e. $u(\cdot,\rho,\mu_1,\ldots,\mu_\gamma)$. If possible, choose the μ_i so that

$$J_1(u(\cdot),y^i(\cdot,\cdot);x_1^i(\cdot)) + \rho\, J_2(u(\cdot),z^i(\cdot,\cdot);x_2^i(\cdot))$$

$$= J_1(u(\cdot),y^j(\cdot,\cdot);x_1^j(\cdot)) + \rho\, J_2(u(\cdot),z^j(\cdot,\cdot);x_2^j(\cdot))$$

for all i, $j\epsilon\{1,..,\gamma\}$. For every $\rho > 0$ for which this can be done, we then have a candidate for a Pareto optimal solution of the problem (1)-(3). It may then be possible to verify that these candidates are solutions by using Theorem 4.1. In checking (a) and (b), of Theorem 4.1, standard sufficiency results for optimal control, such as [10-14], can be used. Of course, choosing γ, $y^i(\cdot,\cdot)$ and $z^i(\cdot,\cdot)$ may be difficult and some ingenuity as well as trial and error must be used.

The general idea of the procedure is to let α_1 take on all possible values in the interval (0,1). For every value of α_1 in this interval we hope to get a Pareto optimal solution. If there is an open loop saddle point solution corresponding to a particular α_1, we use Theorem 3.1; if not, we find a minmax solution and use Theorem 4.1.

6. EXAMPLE

Consider the following problem with scalar $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$

$$J_1(u(\cdot),v(\cdot);x(\cdot)) = x^2(1), \quad J_2(u(\cdot),v(\cdot);x(\cdot)) = -x(1) \quad (20)$$

$$\dot{x}(t) = u(t)v(t), \qquad x(0) = \tfrac{1}{2} \quad (21)$$

$$U = \{u : u^2 \le 1\}\ ,\quad V = \{v : 1 \le v \le 2\} \tag{22}$$

First we form the two player differential game (4)-(6)

$$\hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot);x_1(\cdot),x_2(\cdot)) = \alpha_1 x_1^2(1) - \alpha_2 x_2(1)$$

$$\dot{x}_1(t) = u(t)v_1(t)\ ;\qquad x_1(0) = \tfrac{1}{2}$$

$$\dot{x}_2(t) = u(t)v_2(t)\ ;\qquad x_2(0) = \tfrac{1}{2}$$

Checking for open loop saddle solutions as α_1 ranges between 0 and 1, we find that there are such solutions for $\alpha_1 \varepsilon (0,\ 6/7)$ and that the controls $u(\cdot)$, $v_1(\cdot)$ and $v_2(\cdot)$ are constant. These results are summarized in Table 1.

TABLE 1

α	$\hat{u}(\cdot)$	$\hat{v}_1(\cdot)$	$\hat{v}_2(\cdot)$
$0 < \alpha_1 \le \frac{1}{11}$	1	2	1
$\frac{1}{11} < \alpha_1 \le \frac{1}{3}$	$\frac{1 - 2\alpha_1}{8\alpha_1}$	2	1
$\frac{1}{3} < \alpha_1 \le \frac{1}{2}$	0	$\frac{1 - \alpha_1}{\alpha_1}$	1
$\frac{1}{2} < \alpha_1 \le \frac{2}{3}$	0	$\frac{2(1 - \alpha_1)}{\alpha_1}$	2
$\frac{2}{3} < \alpha_1 \le \frac{6}{7}$	$\frac{1 - \frac{3}{2}\alpha_1}{\alpha_1}$	1	2

The inequality $\hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot)) \le \hat{J}(u(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot))$ was verified via the direct sufficient conditions of [13,14] while the inequality $\hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot)) \ge \hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),v_2(\cdot))$ was established using field type sufficiency theorems [10,11]. That the $\hat{u}(\cdot)$ part of the saddle point solution is a Pareto optimal solution follows from Theorem 3.1.

For $6/7 < \alpha_1 < 1$, there is no open loop saddle point solution and we must resort to Theorem 4.2. To this end, we consider the control problem given by (11)-(12) with $\gamma=2$, $y^1(\cdot) = 1$, $y^2(\cdot) = 1$, $z^1(\cdot) = 2$, $z^2(\cdot) = 2$, $0 < \rho < \frac{1}{6}$, $\mu_1 = 2\rho + \frac{2}{3}$ and $\mu_2 = 1-\mu_1$.

$$K(u(\cdot),y^i(\cdot),z^i(\cdot);x_1^i(\cdot),x_2^i(\cdot)) = \mu_1[x_1^1(1)]^2 + \mu_2[x_1^2(1)]^2 - \rho\mu_1 x_2^1(1) - \rho\mu_2 x_2^2(1)$$

$$\dot{x}_1^1(t) = u(t)\quad ,\qquad \dot{x}_2^1(t) = 2u(t)$$

$$\dot{x}_1^2(t) = 2u(t)\quad ,\qquad \dot{x}_2^2(t) = 2u(t)$$

$$x_1^1(0) = x_1^2(0) = x_2^1(0) = x_2^2(0) = \tfrac{1}{2}$$

Since $u^*(t) = -1/3$ is a solution to this optimal control problem and since $y^i(\cdot) \varepsilon\, \mathcal{L}_1(u^*(\cdot))$, $z^i(\cdot) \varepsilon\, \mathcal{L}_2(u^*(\cdot))$, $i=1,2$, $u^*(\cdot)$ is a Pareto optimal

solution for this problem. Also, since $\rho = \alpha_2/\alpha_1$ and $0 < \rho < 1/6$, this solution corresponds to $6/7 < \alpha_1 < 1$. Note that Theorem 3.1 does not apply when $6/7 < \alpha_1 < 1$ since there is no saddle point solution and Theorem 4.1 is needed.

REFERENCES

1. Goffin, J. and Haurie, A., "Necessary Conditions and Sufficient Conditions for Pareto Optimality in a Multicriterion Perturbed System," Proceedings of the 5th IFIP Conference on Optimization, 1973.

2. Goffin, J. "Conditions for Pareto Optimality in a Multicriterion System with Nondifferentiable Functions," Proceedings of the 11th Annual Allerton Conference on Circuits and System Theory, 1973.

3. Schmitendorf, W. E. "Static Multicriteria Problems: Necessary Conditions and Sufficient Conditions," Proceedings IFAC Symposium on Large Scale Systems, Udine, Italy, June 16-20, 1976.

4. Schmitendorf, W. E. and Moriarty, G. "A Sufficiency Condition for Coalitive Pareto-Optimal Solutions," J. of Opt. Theory and Applications, Vol. 18, No. 1, 1976.

5. Bryson, A. E. and Ho, Y. C. Applied Optimal Control, Blaisdell, Waltham, Mass. 1969.

6. Schmitendorf, W. E. "Existence of Optimal Open-Loop Strategies for a Class of Differential Games," J. of Opt. Theory and Applications, Vol. 5, No. 5, 1970.

7. Schmitendorf, W. E. "Differential Games with Open-Loop Saddle Point Conditions," IEEE Trans. Auto. Control, Vol. AC-15, No. 3, 1970.

8. Schmitendorf, W. E. "A Sufficient Condition for Minmax Control of Systems with Uncertainty in the State Equations," IEEE Trans. Auto. Control, August, 1976.

9. Schmitendorf, W. E. "A General Sufficiency Condition for Minmax Control," (to be submitted for publication).

10. Leitmann, G. "Sufficiency Theorems for Optimal Control," J.Opt. Theory and Application, Vol. 2, pp. 285-292, 1968.

11. Leitmann, G. "A Note on a Sufficiency Theorem for Optimal Control," J. Opt. Theory and Application, Vol. 3, pp. 76-78, 1969.

12. Stalford, H. "Sufficient Conditions for Optimal Control with State and Control Constraints," J. Opt. Theory and Application, Vol. 7, pp. 118-135, 1971.

13. Mangasarian, O. L. "Sufficient Conditions for the Optimal Control of Nonlinear Systems," SIAM J. on Control, Vol. 4, No. 1, 1966.

14. Leitmann, G., and Stalford, H. "A Sufficiency Theorem for Optimal Control," J. of Opt. Theory and Applications, Vol. 8, No. 3, 1971.

19

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFOSR-TR-78-0024	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): 6 MULTICRITERIA OPTIMIZATION WITH UNCERTAINTY IN THE DYNAMICS.	5. TYPE OF REPORT & PERIOD COVERED: 9 Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 W. E. Schmitendorf	8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR 76-2923
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Northwestern University, Dept of Mechanical Eng & Astronautical Sci, Evanston, IL 60201	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304 17 A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Rsch/NM, Bolling AFB, DC 20332	12. REPORT DATE: 11 1977
	13. NUMBER OF PAGES: 8 12 9p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Optimal control problems with a vector performance index and uncertainty in the state equations are investigated. It is assumed that nature chooses the uncertainty, subject to bounds, to maximize the performance index which the controller attempts to minimize. Using Pareto optimality as the optimality criterion, sufficient conditions for an optimal solution are presented. The conditions also suggest a technique for determining the optimal control. The results are illustrated with an example.

403 502 JOB